Electricity and magnetism 2

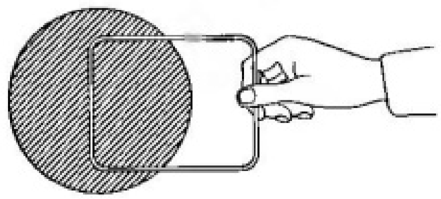
Instructor: A.M. van den Berg

You don't have to use separate sheets for every question.
Write your name and S number on every sheet
There are **6 questions** with a total number of marks: 72

WRITE CLEARLY

(1) (Total 15 marks)

The shaded region represents the pole of an electromagnet where there is a strong magnetic field with strength B perpendicular to the plane of the paper. The rectangular frame with width h is made of 5 mm diameter aluminum. The total length of this frame is given as L.
Suppose that a steady force F of 1 N can pull the frame out in 1 s.



(a) (9 marks)
Derive an equation for the force F in terms of B, v, and the properties of the rectangular frame: wire diameter d (or cross sectional area πr^2), length L, width h, and specific resistivity ρ.

(b) (2 marks)
Then, if the force is doubled, how long does it take to pull the frame out? Explain your answer.

(c) (2 marks)
If the frame is made of brass, with about twice the resistivity, what force is needed to pull it out in 1 s? Explain your answer.

(d) (2 marks)
If the frame is made of 1 cm diameter aluminum, what force is needed to pull out in 1 s? Explain your answer.

N.B.: Neglect inertia of the frame and assume it moves with **constant velocity** v.

(2) (Total 12 marks)
Consider the following electromagnetic wave in free space:

$$E_x = 0 \qquad E_y = E_0 \sin(kx + \omega t) \qquad E_z = 0$$
$$B_x = 0 \qquad B_y = 0 \qquad B_z = -\frac{E_0}{c}\sin(kx + \omega t)$$

(a) (2 marks)
What is the direction of the polarization of this wave?

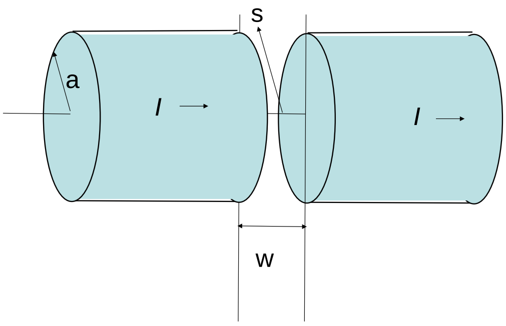
(b) (2 marks)
In which direction does the wave travel?

(c) (4 marks)
Show that this field can satisfy Maxwell's equations if ω and k are related in a certain way.

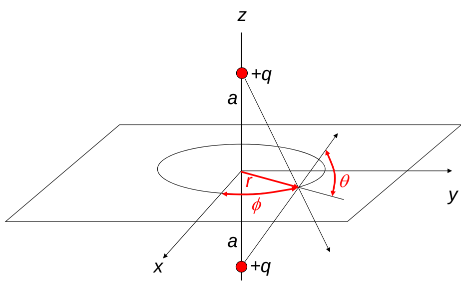
(d) (2 marks)
Suppose $\omega = 10^{10}$ Hz. What is the wavelength?

(e) (2 marks)
Assume that the amplitude $E_0 = 0.5$ V m^{-1}. What is the energy density, averaged over a large region? I.e. calculate the time-averaged energy density.

(3) (Total 10 marks)
A fat wire, with radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in the figure (though not to scale!!). Find the magnetic field in the gap at a distance $s < a$ from the axis.



(4) (Total 11 marks)
Consider two equal point charges q, separated by a distance $2a$; see the figure. Construct the xy plane equidistant from the two charges. By integrating the Maxwell stress tensor over this plane, determine the force of one charge on the other, where the force is given as:



$$\vec{F} = \oint_{\mathcal{S}} \overleftrightarrow{T} \circ d\vec{a} - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \vec{S} d\tau$$

The components of the Maxwell stress tensor $\overleftrightarrow{T}$ are given as:

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)$$

(a) (3 marks)
Give the three components of $\vec{E} = (E_x, E_y, E_z)$, $\vec{B} = (B_x, B_y, B_z)$, and $\vec{S} = (S_x, S_y, S_z)$ in terms of q, r, ϕ, and θ for this configuration in the equidistant plane. Here θ and ϕ are the angles as indicated in the figure.

(b) (3 marks)
Give an expression for the components of $d\vec{a} = (da_x, da_y, da_z)$ in terms of r, ϕ, and θ.

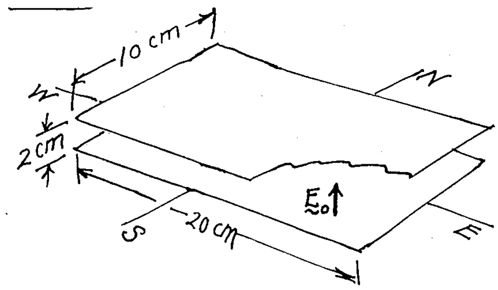
(c) (5 marks)
Determine the force on the upper charge using the Maxwell stress tensor.

Hint: you may want to use

$$\int_0^\infty \frac{r^3 dr}{(r^2 + a^2)^3} = \frac{1}{4a^2}$$

(5) (Total 16 marks)

A capacitor consists of two parallel rectangular plates with a vertical separation of 2 cm. The East-West dimension of the plates is 20 cm, the North-South dimension is 10 cm. The capacitor has been charged by connecting it temporarily to a battery of 300 V. The direction of the electric field vector is pointing upwards; see the figure.



(a) (2 marks)
What is the electric field strength between these plates?

(b) (2 marks)
How many excess electrons are on the negative plate?

Now give the following quantities as they would be measured in a frame of reference which is moving Eastward, relative to the laboratory in which the plates are at rest, with speed 0.6 c.

(c) (2 marks)
The three dimensions of the capacitor.

(d) (2 marks)
The number of excess electrons on the negative plate.

(e) (2 marks)
The electric field strength between the plates.

(f) (2 marks)
The magnetic field strength between the plates.

(g) (2 marks)
What is the direction of this magnetic field?

(h) (2 marks)
What will be the electric field strength between the plates for a frame of a reference which is moving up with speed 0.6?

(6) (Total 8 marks)

A plane wave traveling through vacuum in the positive z direction encounters a perfect conductor, occupying the region ($z \geq 0$), and reflects back. The electric field vector is then given as:

$$\vec{E}(z,t) = E_0 \left[\cos(kz - \omega t) - \cos(kz + \omega t)\right] \hat{x} \quad (z < 0)$$

(a) (2 marks)
Explain why this expression provides a valid description of the wave equation for $z < 0$.

(b) (2 marks)
Find the accompanying magnetic field in the region $z < 0$.

(c) (2 marks)
Assuming $\vec{B} = \vec{0}$ inside the conductor, find the surface current K on the interface $z = 0$, by invoking the proper boundary condition (see below).

(d) (2 marks)
Find the time-averaged magnetic force per unit area on the surface.

Hint: at the interface between two mediums 1 and 2, the relations between the electromagnetic fields are given as:

$$\begin{aligned} \epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} &= \sigma_f \\ B_1^{\perp} - B_2^{\perp} &= 0 \end{aligned} \qquad \begin{aligned} E_1^{\parallel} - E_2^{\parallel} &= 0 \\ \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} &= \vec{K}_f \times \hat{n} \end{aligned}$$

where $\hat{n}$ is the normal to the interface.